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On the existence of monotonic fronts for a class of physical problems described by the equation $\lambda w''' + w' = f(w)$

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Abstract. We obtain an upper bound on the value of λ for which monotonic front solutions of the equation $\lambda w''' + w' = f(w)$ with $\lambda > 0$ may exist.

1. Introduction

In a variety of physical phenomena the structure of fronts is described by a third-order differential equation of the form

$$\lambda w''' + w' = f(w) \qquad \lambda > 0 \quad x \in \mathbb{R} \tag{1.1}$$

where primes denote derivatives with respect to x, f is a positive and continuous function for $w \in (-1, 1)$ and such that $f(-1) = f(1) = 0$. For example, equation (1.1) with $f(w) = \cos(\pi w/2)$ and λ small arises in the geometric model of crystal growth [1, 2]. A more complicated version of the geometric model of crystal growth is given by equation (1.1) with $f(w) = \cos(\pi w/2)/(1 + \alpha\cos(2\pi w))$, where $0 < \alpha < 1$ represents crystalline anisotropy. Traveling wave solutions of the Kuramoto–Sivashinsky equation which arises in the study of reaction–diffusion systems [3], flame propagation [4], and others, obey the above equation with $f(w) = 1 - w^2$. In this latter case $\lambda = (c/2)^2$ where c is the speed of the travelling wave. Our aim in this paper is to determine generic bounds on λ for which equation (1.1) has no monotonic fronts; i.e. has no solutions w with $w' > 0$ and such that $\lim_{x\to-\infty} w = -1$, $\lim_{x\to\infty} w = 1$. For the case $f(w) = 1 - w^2$ bounds of this sort were found by Toland [5]. In fact, he proved that for $\lambda \geqslant \frac{2}{9}$ there is no monotonic solution of (1.1) on $\mathbb{R}$. Although Toland's bound is certainly correct, for the case in question it is now known that for all $\lambda > 0$ there is no monotonic solution of (1.1) (see, e.g., [6, 7]). For the equation describing needle crystals (i.e. where $f(w) = \cos(\pi w/2)$) it has also been shown that no monotonic solutions exist [8, 9]. In spite of these negative results, there are explicit examples of functions f for which monotonic fronts do exist. This is the case for the modified equation of the geometric model of crystal growth (i.e. for $f(w) = \cos(\pi w/2)/[1 + \alpha\cos(2\pi w)]$) for a discrete set of values of the crystal anisotropy parameter α [10]. A simpler example for which monotonic fronts exist is given by $f(w) = \frac{1}{2}(1 - w^2)(1 - \frac{\lambda}{2} + \frac{3}{2}\lambda w^2)$, for which the front $w(x) = (e^x - 1)/(1 + e^x) = \tanh(\frac{x}{2})$ is monotonic, satisfies equation (1.1) for this f and also the boundary values $\lim_{x\to\pm\infty} w = \pm 1$. Moreover, one can construct many other explicit examples of f's for which equation (1.1) exhibits monotonic fronts. Here we prove a generic bound on the values of λ for which equation (1.1) together with the boundary values $\lim_{x\to\pm\infty} w = \pm 1$ does not have monotonic solutions. Our main result is as follows.

Theorem. If

$$\lambda > 0.2288\left(\int_{-1}^{1} f(t)(1-t^2)\,\mathrm{d}t\right)^{-2} \tag{1.2}$$

then there is no solution of

$$\lambda w''' + w' = f(w)$$

satisfying $\lim_{x\to\pm\infty} w = \pm 1$ and $w' \geqslant 0$ on $\mathbb{R}$.

Several remarks are in order concerning this result. First, for the case considered by Toland, that is for $f(w) = 1 - w^2$, our bound is slightly better than his ($\lambda_{\mathrm{Toland}} = \frac{2}{9} \approx 0.222$, $\lambda_{\mathrm{here}} = 0.201$), although we know that both of these bounds are not relevant because of the non-existence results of Jones *et al* [6] (see also [7, 11]). Second, our bound is not optimal, in the sense that there is no f for which the inequality (1.2) is saturated (i.e. satisfied as an inequality). Third, the methods used here to prove bounds on λ for which there are no monotonic solutions can easily be extended to treat more general equations. In particular they have been used by us to determine bounds on the speed of monotonic travelling fronts of a Kuramoto–Sivashinsky equation with dispersion [12]. We do not attempt to prove the existence of fronts, which requires an entirely different approach [5]. The rest of the paper is organized as follows: in section 2 we prove the bound and in section 3 we apply our bound to several examples.

2. Proof of the bound on λ

Here we are only concerned about monotonic solutions $w(x)$ of equation (1) satisfying $w(x) \to -1$ as $x \to -\infty$ and $w(x) \to +1$ as $x \to +\infty$. In view of this, it is convenient to consider the dependence of the independent variable x as a function of w, or rather the dependence of $u(w) \equiv (\mathrm{d}x/\mathrm{d}w)^{-1}$ as a function of w. In fact, for a monotonic solution $w(x)$ of (1), $x(w)$ increases monotonically from $-\infty$ to $+\infty$ as w goes from -1 to $+1$. Thus, the function $u(w)$ is non-negative and vanishes at both ends. Since the original equation (1.1) is autonomous, one can rewrite it as a second-order equation for $u(w)$. In terms of u, $\mathrm{d}w/\mathrm{d}x = u$, $\mathrm{d}^2w/\mathrm{d}x^2 = u\mathrm{d}u/\mathrm{d}w$ and $\mathrm{d}^3w/\mathrm{d}x^3 = \frac{1}{2}u\mathrm{d}^2(u^2)/\mathrm{d}w^2$. Therefore, equation (1.1) can be rewritten as

$$\frac{1}{2}\lambda u\frac{\mathrm{d}^2u^2}{\mathrm{d}w^2} + u = f(w) \qquad w \in (-1, 1) \tag{2.1}$$

together with the boundary condition $u(-1) = u(+1) = 0$. This is a nonlinear second-order differential equation for $u(w)$ that is singular at both end-points.

In order to prove the desired bound on λ, we multiply (2.1) by $g(w)/u$, where $g(w)$ is any continuous function such that $g(w)$ is twice differentiable, $g(\pm 1) = 0$ and $g(w)$ is concave (i.e. $-g'' \geqslant 0$), therefore g is positive. A specific choice for g will be made shortly. Hence we have

$$\frac{\lambda}{2}g(w)\frac{\mathrm{d}^2(u^2)}{\mathrm{d}w^2} + g(w) = \frac{f(w)}{u}g(w)\,. \tag{2.2}$$

We now integrate (2.2) in w between -1 and 1. After integrating the first term on the left-hand side by parts we obtain

$$\frac{\lambda}{2}\int_{-1}^{1} g''u^2\,\mathrm{d}w + \int_{-1}^{1} g(w)\,\mathrm{d}w = \int_{-1}^{1}\frac{f(w)}{u}g(w)\,. \tag{2.3}$$

Note that when integrating by parts we have used the fact that both g and u vanish at the end-points. Let $h = -g''$. Since g is concave, h is positive. From (2.3) we have

$$\int_{-1}^{1} g(w)\,\mathrm{d}w = \int_{-1}^{1} \left(\frac{f(w)}{u} g(w) + \frac{\lambda}{2} h u^2 \right) \mathrm{d}w\,. \tag{2.4}$$

Since f, g and h are positive in $(-1, 1)$ and λ is a positive constant, for any fixed w we have

$$\frac{f(w)g(w)}{u} + \frac{\lambda}{2} h(w) u^2 \geqslant \frac{3}{2} (fg)^{2/3} \lambda^{1/3} h^{1/3} \tag{2.5}$$

(just minimize the right-hand side as a function of u for $u \in (0, +\infty)$). From (2.4) and (2.5) we have

$$\lambda^{1/3} \leqslant \frac{2}{3} \int_{-1}^{1} g(w)\,\mathrm{d}w \left(\int_{-1}^{1} (fg)^{2/3} h^{1/3}\,\mathrm{d}w \right)^{-1}. \tag{2.6}$$

The bound on λ given by (2.6) holds for any function g twice differentiable in $(-1, 1)$ such that $h = -g'' \geqslant 0$ and $g(\pm 1) = 0$. If λ is larger than the right-hand side of (2.6) for fixed f and any such g, equation (1) cannot have monotonic fronts. For explicit examples of f's one can use directly (2.6) to derive upper bounds on λ. However, here we would like to express a bound on λ solely in terms of f (i.e. an explicit generic bound on λ). It is for this reason that we will pick a specific g in order to prove our main result. So choose g in such a way that $h = -g'' = f$ in $(-1, 1)$ and $g(\pm 1) = 0$. Such a g can be written explicitly in terms of f as

$$\int_{-1}^{1} K(s, t) f(t)\,\mathrm{d}t \tag{2.7}$$

with $K(s,t) = \frac{1}{2}(s+1)(1-t)$ for $-1 \leqslant s < t$ and $K(s,t) = \frac{1}{2}(1+t)(1-s)$ for $t < s \leqslant 1$. With this particular choice of g, the bound (2.6) can be expressed as

$$\lambda^{1/3} \leqslant \frac{2}{3} \int_{-1}^{1} g(w)\,\mathrm{d}w \left(\int_{-1}^{1} f g^{2/3}\,\mathrm{d}w \right)^{-1} = \frac{2}{3} \int_{-1}^{1} g\,\mathrm{d}w \left(\int_{-1}^{1} (-g'') g^{2/3}\,\mathrm{d}w \right)^{-1} \tag{2.8}$$

and integrating the denominator of the right-hand side of (2.8) by parts we get

$$\lambda^{1/3} \leqslant \int_{-1}^{1} g\,\mathrm{d}w \left(\int_{-1}^{1} (g')^2 g^{-1/3}\,\mathrm{d}w \right)^{-1}. \tag{2.9}$$

Writing $g = \psi^{6/5}$, the denominator $\int_{-1}^{1} (g')^2 g^{-1/3}\,\mathrm{d}w$ becomes $\frac{36}{25} \int \psi'^2$. Therefore

$$\lambda^{1/3} \leqslant \frac{25}{36} \int_{-1}^{1} \psi^{6/5}\,\mathrm{d}w \left(\int_{-1}^{1} (\psi')^2\,\mathrm{d}w \right)^{-1}. \tag{2.10}$$

Let I denote the maximum of the quotient $R(\Phi) = (\int_{-1}^{1} \Phi^{6/5}\,\mathrm{d}w)^{5/3} / \int_{-1}^{1} (\Phi')^2\,\mathrm{d}w$ taken over all functions $\Phi \in C^1(-1, 1)$ (to be precise, the maximum of the quotient is taken over all functions Φ in the Sobolev space $H_0^1(-1, 1)$). Clearly, $R(\Phi)$ is homogeneous in Φ. Then equation (2.10) may be written as

$$\lambda^{1/3} \leqslant \frac{25}{36} \left(\int_{-1}^{1} \psi^{6/5}\,\mathrm{d}w \right)^{-2/3} R(\psi)\,. \tag{2.11}$$

Since $R(\psi) \leqslant I$, and using $g = \psi^{6/5}$, equation (2.11) implies

$$\lambda^{1/3} \leqslant \frac{25}{36} I \left(\int_{-1}^{1} g\,\mathrm{d}w \right)^{-2/3}. \tag{2.12}$$

It is not difficult to show that the maximum I of $R(\Phi)$ in $H_0^1(-1,1)$ exists and that the corresponding maximizing function is unique up to a multiplicative constant. The maximum in $H_0^1(-1,1)$ is attained by a function ϕ being in fact C^∞ in the open interval and satisfying the following differential equation:

$$-\phi'' = \phi^{1/5} \qquad \text{in } (-1,1) \tag{2.13}$$

together with the boundary conditions $\phi(-1) = \phi(1) = 0$. One can solve numerically (2.13) and evaluate $I = (\int_{-1}^{1} \phi^{6/5}\,\mathrm{d}w)^{5/3} / \int_{-1}^{1} \phi(-\phi'')\,\mathrm{d}w = (\int_{-1}^{1} \phi^{6/5})^{2/3}$. The numerical value of I is approximately 0.5548. From equation (2.12) we have

$$\lambda^{1/3} \leqslant 0.3853\left(\int_{-1}^{1} g\,\mathrm{d}w\right)^{-2/3} \tag{2.14}$$

Using (2.7) we can evaluate $\int_{-1}^{1} g\,\mathrm{d}w$ explicitly in terms of f. We have $\int_{-1}^{1} g\,\mathrm{d}w = \int_{-1}^{1}\int_{-1}^{1} K(w,t) f(t)\,\mathrm{d}t\,\mathrm{d}w = \int_{-1}^{1} f(t)\{\int_{-1}^{t} K(w,t)\,\mathrm{d}w + \int_{t}^{1} K(w,t)\,\mathrm{d}w\}\,\mathrm{d}t = \frac{1}{2}\int_{-1}^{1} f(t)(1-t^2)\,\mathrm{d}t$ so finally we get our bound

$$\lambda \leqslant 0.2288\left[\int_{-1}^{1} f(t)(1-t^2)\,\mathrm{d}t\right]^{-2}. \tag{2.15}$$

Hence if, for a given f, λ is larger than the right-hand side of (2.13), equation (1.1) has no monotonic fronts.

3. Applications

We first consider the equation for needle crystals including anisotropy. This corresponds to our equation (1.1) with

$$f(w) = \cos(\pi w/2)/(1 + \alpha\cos(2\pi w)) \qquad 0 < \alpha < 1. \tag{3.1}$$

In this case it has been shown [10] that monotonic fronts exist for a discrete set of values of α and small λ. This f vanishes at $w = \pm 1$ and, for $0 < \alpha < 1$, f is positive so our theorem applies here.

If we insert $f(w)$ given by (3.1) in equation (2.14) we get an upper bound $\lambda_u(\alpha)$ on the possible values of λ for which one could have monotonic fronts. This function $\lambda_u(\alpha)$ is shown in figure 1. Note that $\lambda_u(\alpha)$ is decreasing, $\lambda_u(0) = 0.214$ and $\lambda_u(1) = 0$.

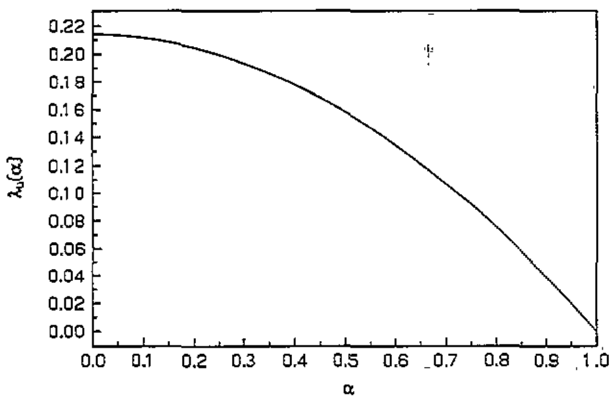

Figure 1. Upper bound on the value of λ for the existence of monotonic fronts in the geometric model of crystal growth with anisotropy.

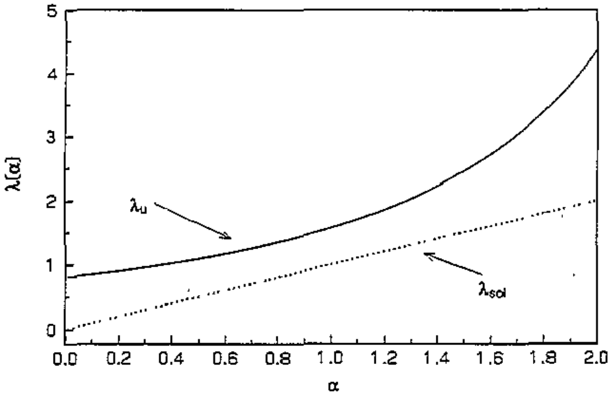

Figure 2. The solid line depicts the upper bound on the value of λ for the existence of fronts of the exactly solvable example. The dotted line corresponds to the values for which it is known that there is a solution.

As a second example we consider an exactly solvable model given by equation (1.1) with

$$f(w) = \tfrac{1}{2}(1 - w^2)(1 - \tfrac{\alpha}{2} + \tfrac{3}{2}\alpha w^2) \qquad 0 < \alpha < 2\,. \tag{3.2}$$

In this case, monotonic fronts exist when $\lambda = \alpha$. In fact the solution of equation (1.1) with f given by (3.2) and $\lambda = \alpha$ is given by $w(x) = \tanh(\frac{x}{2})$. The function f given by (3.2) vanishes at $w = \pm 1$, and for $0 < \alpha < 2$ it is positive, so again in this case our theorem applies. Inserting (3.2) in (2.14) we get an explicit bound $\lambda_u(\alpha)$ given by

$$\lambda_u(\alpha) = \frac{39.2765}{(7 - 2\alpha)^2}\,.$$

In figure 2 we have plotted this bound. The solid line corresponds to $\lambda_u(\alpha)$ while the dotted line corresponds to $\lambda = \alpha$, the exact value for which it is known that there is a front.

As a final remark, we wish to point out that, if in a particular case a better bound is sought, one may go back to equation (2.6) and find the best g for the problem. The method presented here can also be used in equations of the form $\lambda w''' + w'' + w' = f(w)$, with $\lambda > 0$, $f(\pm 1) = 0$ and f positive and continuous between -1 and 1. In order to get a bound for this equation an adequate choice for the trial function g has to be made. The choice depends on f. Some results for $f(w) = 1 - w^2$ are given in [12].

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